Optimization on Manifolds, Appendix: Functions

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1 Definition

There is no generally accepted manner for defining a function, it's domain, or it's codomain, thus this section is rather used as a means to denote the book's conventions for how these terms are defined. Furthermore to provide a more thorough introdution into functions there is a more abstract definition, **morphisms**.

A **function**, f, is a mapping from one set to another set,

$$f : A \to B$$

such that f is a set of ordered pairs (a, b) where $a \in A$ and $b \in B$ and has the condition that if $(a, b) \in f$ and $(a, c) \in f$ then $b = c$. This means that an element in the set A may only be the 1^{st} element of 1 ordered pair. Otherwise the function would be ambigious allowing an input to map to multiple outputs. If $(a, b) \in f$ then we can say $b = f(a)$. Note that $f(a)$ need not be defined for all $a \in A$ in this book. Thus this definition of a function aligns more with the concept of **partial functions** which may be seen as a generalization of a function by not requiring it to map every element of A to B.

The **domain** of a function f, is also denoted as the input and can be defined as the elements in A such that there exists an element in B in order for the ordered pair (a, b) to be in f.

$$dom(f) := \{a \in A : \exists b \in B : (a, b) \in f\}$$

The **range** of a function f, also known as the image can be seen as the resulting set based on applying the function to the elements in A.

$$range(f) := \{b \in B : \exists a \in A : (a, b) \in f\}$$

The **pre-image** of a function f we say is $f(Y)^{-1}$ and is defined as $f(Y)^{-1} := \{a \in A : f(a) \in Y : Y \subseteq B\}$. This means that the pre-image is a subset of the domain.

A **surjective** function is one that maps *onto* a set. More specifically for every element $b \in B$ there must exists an element $a \in A$ such that $f(a) = b$. This is equivalent to saying that a functions range is equivalent to the codomain.

$$\forall b \in B \;\; \exists a \in A : f(a) = b$$

A **codomain** is the set that f can map to similar to the domain which is the set that f maps from.

An **injective** function f such that if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$. Simple examples for visualization of injective and non-injective would be $x \mapsto x$ and $x \mapsto x^2$, respectively. A **bijective** function is one that is both surjective and injective and thus has a one-to-one correspondence between the sets. Note, that in this book, the codomain and range are not differentiated (commonly done in set-theory or graph definitions) and thus a bijective function need only be injective since every function is surjective.

A **morphism** is a structure-preserving map between algebraic structures. That means for set theory it is functions, linear algebra uses linear transformations, topology uses continuous functions, etc.

A **homomorphism** is a morphism between algebraic structure of the same type.

An **isomorphism** is a homomorphism that admits an inverse mapping. This means that $f : A \to B$ is an isomorphism if and only if there exists a function g such that if $f(a) = b$ then $g(b) = a$. For a morphism/function f to admit an inverse f must be bijective as otherwise the inverse function g would not hold the original property nor would g be on the set B.

A **homeomorphism** is an isomorphism that is continuous. For topological spaces a definition of a **continuous function** f is a function such that for every open set $V \in B$ the pre-image is an open set in A.

A **diffeomorphism** is an isomorphism between smooth manifolds. Smooth simply means that the structure belongs to C^∞ and hence can be differentiated infinitely many times. This is not to be confused with a **differentiable manifold** which is a generalization of a smooth manifold.