Optimization on Manifolds, Appendix: Topology

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1 Definitions

This serves as a light introduction into the topic of topology, but is not sufficient to have a complete or deep understanding. For further information, I suggest reading James Munkres' *Topology* text for a thorough introduction.

Just as an n-dimensional vector space is an abstraction of $\mathbb{R}^n$, a topology on a set X is an abstraction of open sets in $\mathbb{R}^n$.

A **topology** on a set X is a collection $\mathcal{T}$ of open subsets of X such that:

1. $X, \{0\} \in \mathcal{T}$
2. Given a subcollection $\mathcal{S}$ of $\mathcal{T}$, $\cup_i S_i \in \mathcal{T}$
3. Given a finite subcollection $\mathcal{S}$ of $\mathcal{T}$, $\cap_i S_i \in \mathcal{T}$

Thus a topology is the couple $(X, \mathcal{T})$, but when notation allows it will be abbreviated to X. Currently, the definition for topology is not well-defined as we do not know what "open" means.

Let X be a topological space. Let A be a subset of X. A **neighborhood** of a point x on X is a subset $\mathcal{V}$ that includes an open set, $\mathcal{U}$ containing x.

$$x \in \mathcal{U} \subset \mathcal{V} \subseteq X$$

A **limit point** or **accumulation point** of a subset A of X is a point x of X such that every neighborhood of x intersects A in some point *other than* x. Note that it is important to distinguish between intersecting at x and somewhere else. If the restriction of intersecting at points other than were to be removed x would be a **point of closure**. Thus every point of closure is a limit point but not necessarily every limit point is a point of closure.

A subset is **closed** if and only if it contains all it's limit points. An open set is naturally one that does not contain all it's limit points.

A sequence of points $\{x_k\}_{k=1,2,...}$ of X **converges** to a point $x \in X$ if for every neighborhood $\mathcal{U}$ of x there exists a positive integer K such that x_k belongs to $\mathcal{U}$ for all $k \geq K$.

Since topology needs to satisfy relatively few axioms it is natural that properties that may hold for $\mathbb{R}^n$ may not hold for X. As an example a **singleton** which is a set containing only one element may not be closed in the topological sense. Take for example the topology of $[-1, 1]$. The open sets are $[-1, a)$ for $a > 0$,(b, a) for $a < 0, b > 0$, and $(b, 1]$ for $b < 0$ *(Why?)*. To avoid such situations **separation axioms** have been introduced to make a distinction between topologies.

A topological space X is T_1; **accessible** or **Fréchet**, if for any distinct points x and y of X there is an open set containing x and not y. (Every singleton is closed.)

A topological space X is T_2, **Hausdorff**, if any two distinct points of X have disjoint neighborhoods. A Hausdorff topology means that any sequence of points on X converges to at most one point of X.*(Why?)*

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topological spaces of X. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$ then $\mathcal{T}_2$ is said to be **finer**.

A **base** or **basis** for a topology on set X is a collection $\mathcal{B}$ of subsets of X such that

1. each $x \in X$ belongs to at least one element in $\mathcal{B}$
2. if $x \in (B_1 \cap B_2)$ with $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ is a base for topology $\mathcal{T}$ then $\cup_i \mathcal{B} = \mathcal{T}$ and a topology is denoted as **second-countable** if it's base is finite.

If X and Y are topological spaces then the **product topology** $X \times Y$ has the base $\mathcal{B}$ which is the collection of all sets of the form $U \times V$ where U is an open set of X and V is an open set of Y.

If Y is a subset of a topological space $(X, \mathcal{T})$. Then $\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$ is a topology on Y called the **subspace topology**.

A collection $\mathcal{A}$ of subsets of X **cover** or is a **covering** of X if the union of the elements of $\mathcal{A}$ is equal to X.

Heine-Borel Theorem: A subset of $\mathbb{R}^n$ with the subspace topology is compact if and only if it is closed and bounded.